THE VALIDITY OF THE PSYCHOPHYSICAL LAW FOR THE ESTIMATION OF SURFACE MAGNITUDES.

By J. McCrea, B. A., and H. J. Pritchard, B. A. (Toronto).

The Psychophysical Law has been tested as to its application, for linear magnitudes, but hitherto little has been done to find out to what extent it may be applied to the estimation of surfaces. This paper gives the results of a series of experiments which have been made for the purpose of testing the validity of the law when applied to the estimation of surface magnitudes. The experiments have been performed in the Psychological Laboratory of the University of Toronto, during the session of 1896-97.

The apparatus used, constructed under the direction of Dr. Kirschmann, is essentially that used by Mr. J. O. Quantz, B. A., when experimenting on a problem akin to the present investigation during the session of 1893-94. For a description of this apparatus the readers are referred to Vol. VII, No. 1, of this Journal, to the article on "The Influence of the Color of Surfaces on our Estimation of their Magnitudes." Except for one or two minor changes the arrangement was the same as in Mr. Quantz' experiments. In the present investigation the colored surfaces were replaced by white, which were obtained by simply leaving the surface in the diaphragm open and covering the open end of the apparatus with white tissue papers in addition to the ground glass of the window.

The objects used in our case were taken from a set of 180 brass diaphragms, varying in diameter from 6 to 50 mm., which were constructed according to the device of Dr. Kirschmann, by means of a conical drill which bored through the above stated number of thin sheets of brass, which were tightly clasped together. By this means the necessity of any further finishing, such as grinding or filing, which would have spoiled the continuity of the transitions, was prevented. In order to secure a constant distance of the periphery of the circular diaphragm from one of the margins of the brass plate, the plates were set in such a way that one surface of the parallelopipedon, which they formed, was parallel to the cutting edge of the conical drill.

From this set of diaphragms we selected for each experiment two discs of different size. This was done in order to eliminate the error which might arise from knowing the posi-

tion in which the discs had actually the same visual angle. The difference between their diameters which was most convenient for us to use was about one millimeter. The measurement of the diameters of these was made with a micrometer caliper, by which it was possible to measure distances to $\frac{1}{50}$ of a millimeter. Each observer measured each disc 10 times, and the average of the 20 trials was taken.

As the discs were made with a conical drill, the diameter of the aperture on one side was somewhat smaller than that on the other. The side on which the diameter was the smaller was blackened, so as to prevent the reflection of light as much as possible. Then the diaphragms were placed on the screens, with the blackened side toward the eye of the observer.

With the same micrometer caliper also was ascertained the distance between the two discs when in the same plane. This distance remained approximately constant for all the series. The distance between the fixed or normal disc and the eye of the observer was 1230 mm., and this remained constant throughout.

In the observations, the mode of operation was the same as that described by Mr. Quantz in his paper. Each observer made 100 observations with each eye in every series of experiments, but they were not made consecutively. Only 20 or 30 observations with each eye were made by an observer in succession, and in these he made 10 observations with each eye alternately. This change was made so that the eye would not become wearied by too prolonged use. These 100 observations with each eye are called a series.

After each observer had made 100 observations with each eye, the discs were interchanged and the same mode of procedure was followed. These observations are called the second series and these two series of observation are called a set. As will be seen from the accompanying table, six such sets of experiments have been performed.

Although the intention was to apply the Method of Average Errors, the ordinary course of procedure in the calculation was not followed; for, instead of the pure average error, on the advice of the director of the laboratory, we computed the Mean Variation. The Mean Variation has the advantage of being entirely independent of the normal magnitude, thus giving only the average deviation from an ideal normal magnitude, represented by the average value of the observations. If, to the determination of the normal magnitude, an error of $\frac{1}{10000}$ would adhere, this error would be implied in each observation, and if the observations were all in one direction, either positive or negative, the error would enter the last result multiplied by the number of cases. Further, by using the Mean Variation, one escapes the ambiguity which is

involved in the ordinary use of the Average Error Method, viz.: in the Method of Pure Average Errors, we can change, without detriment to the pure average error, one observation of a series (*e. g.*, from the positive to the negative) without changing any other. This is not possible if we take the mean variation.

In employing this method, it was applied to the distances of the variable stimulus from the normal stimulus. First, the average distance between them was found and the mean variation in the 100 observations computed. Then the average distance was added to, or subtracted from, the 1230 mm., according to the direction from the normal stimulus in which the average distance was found to be. The result thus obtained was called r. The mean variation was added to r, and the result called r^{u}; it was likewise subtracted from r, and the result called r_{l}. Then by means of a trigonometrical solution, the value of the visual angles, subtended by the diameter of the disc at each of these three distances, was ascertained. The computation is exactly the same as in the case of Mr. Quantz, to whose geometrical representations the reader is referred. Then, having found the difference between the angle subtended by the disc at the distance r and each of the other two angles, the average of these two differences was taken, and this average is regarded in our tables as the mean variation. The relation between this variation and the visual angle of the diameter of the disc at the distance r was expressed in percentage; and the averages were taken of the per cent. for both of the eyes in both series of the set.

From this percentage of the mean variation of the diameter, the percentage of the mean variation of the surface magnitude may be deduced. By means of the following algebraic process, it was ascertained that the relation between these two percentages remained constant.

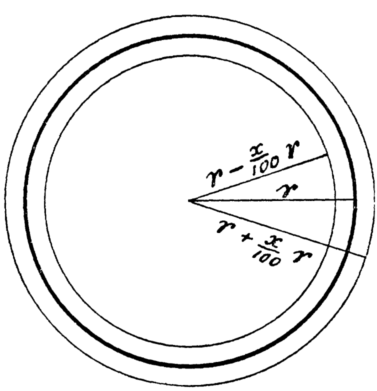

FIG. 1.

$r =$ radius of circle, which represents the estimated value.
$x =$ the % of r, which represents the mean variation.
$(r + \frac{x}{100}\, r)$ and $r - (\frac{x}{100} \cdot r) =$ the radii of the circles between which the variations of the circle have play.

1. Area of Estimated Value $= \pi r^2$
2. Area of inc. circle $= \pi (r + \frac{x}{100}.r)^2 = \pi r^2 (1 + \frac{2x}{100} + \frac{x^2}{10000})$
3. Area of dec. circle $= \pi (r - \frac{x}{100}.r)^2 = \pi r^2 (1 - \frac{2x}{100} + \frac{x^2}{10000})$

Diff. of 2 from estim'd value $= \pi r^2 (1 + \frac{2x}{100} + \frac{x^2}{10000}) - \pi r^2$

$$= \pi r^2 (\frac{2x}{100} + \frac{x^2}{10000})$$

Diff. of 3 from estim'd value $= \pi r^2 - \pi r^2 (1 - \frac{2x}{100} + \frac{x^2}{10000})$

$$= \pi r^2 (\frac{2x}{100} - \frac{x^2}{10000})$$

Av. of these var't'ns $= \dfrac{\pi r^2 (\frac{2x}{100} + \frac{x^2}{10000}) + \pi r^2 (\frac{2x}{100} - \frac{x^2}{10000})}{2}$

$$= \pi r^2 (\frac{2x}{100})$$

$\therefore$ the per cent. of the variation to the estimated value (πr^2) is $2x$%, *i. e.*, the per cent. of the mean variation of the surface magnitude is always double the per cent. of the mean variation of the diameter of estimated value of the circle.

In the tables which we give with this paper, we refer only to the diameter, but if we wish to ascertain the variation of the area, we merely require to multiply the former percentage by 2, as we have seen from the foregoing calculation.

In the experiments which have been performed, six different pairs of discs have been taken. The sizes of these discs are shown in the diagram (Fig. 2) and the results of the experiments are given in the table below.

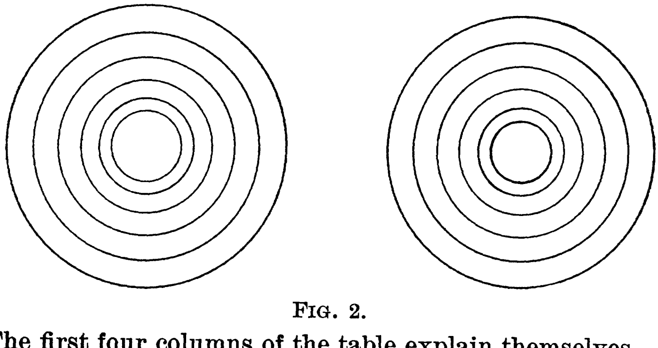

Fig. 2.

The first four columns of the table explain themselves. In the 5th is given the measurement of the diameter of the normal stimulus, *i. e.*, of the disc which remains fixed throughout the series. In the 6th column is given the visual angle

TABLE I.

Set.	Observer.	Series.	Eye.	Measurement of Normal Stimulus.	Visual Angle of Normal Stimulus.	Estimated Value.	Mean Variation.	% of Estimated Value.	Average %.	Constant Error.	Average of Constant Errors
I	McC.	1	Left	35.902 mm.	1° 40′ 14.01″	1° 36′ 34.73″	1′ 18.2″	1.34%	1.3125%	—3′ 39.28″	—1′ 29.1275″
			Right			1 36 13.62	1 12.92	1.26		—4 0.39	
		2	Left	37.306 mm.	1 44 9.26	1 45 14.95	1 28.31	1.24		+1 5.69	
			Right			1 44 46.73	1 28.76	1.41		+0 37.47	
	P.	1	Left	35.902 mm.	1 40 14.01	1 39 35.63	1 1.265	1.02	1.0975	—0 38.38	+1 0.1825
			Right			1 39 23.05	1 2.715	1.05		—0 50.96	
		2	Left	37.306 mm.	1 44 9.26	1 47 37.63	1 22.375	1.27		+3 28.37	
			Right			1 46 10.96	1 7.18	1.05		+2 1.70	
II	McC.	1	Left	30.155 mm.	1° 24′ 12.3″	1° 27′ 30.66″	1′ 11.67″	1.36%	1.2975%	+3′ 18.36″	+0′ 33.3525″
			Right			1 26 50.98	1 7.18	1.29		+2 38.68	
		2	Left	28.958 mm.	1 20 51.67	1 19 10.14	0 56.98	1.19		—1 41.53	
			Right			1 18 49.57	1 4.18	1.35		—2 2.10	
	P.	1	Left	30.155 mm.	1 24 12.3	1 26 56.53	0 56.09	1.07	0.975	+2 44.23	+1 5.5275
			Right			1 26 7.88	0 51.325	0.99		+1 55.58	
		2	Left	28.958 mm.	1 20 51.67	1 20 35.6	0 43.24	0.89		—0 16.07	
			Right			1 20 53.3	0 46.095	0.95		+0 1.63	

TABLE II.

Set.	Observer.	Series.	Eye.	Measurement of Normal Stimulus.	Visual Angle of Normal Stimulus.	Estimated Value.	Mean Variation.	% of Estimated Value.	Average %.	Constant Error.	Average of Constant Errors
III	McC.	1	Left	22.73 mm.	1° 3′ 30.66″	1° 1′ 28.2 ″	0′ 43.4 ″	1.17%	1.3925%	—2′ 2.46″	—0′ 3.6075″
			Right			1 1 25.72	0 50.88	1.38		—2 4.94	
		2	Left	23.923 mm.	1 6 48.58	1 9 6.08	1 6.40	1.60		+2 17.50	
			Right			1 8 24.05	0 58.53	1.42		+1 35.47	
	P.	1	Left	22.73 mm.	1 3 30.66	1 3 2.37	0 43.55	1.15	1.1775	—0 28.29	+0 45.4075
			Right			1 2 53.81	0 43.96	1.16		—0 36.85	
		2	Left	23.923 mm.	1 6 48.58	1 9 24.66	0 54.06	1.29		+2 36.08	
			Right			1 8 19.27	0 45.78	1.11		+1 30.69	
IV	McC.	1	Left	16.245 mm.	0° 45′ 22.68″	0° 45′ 9.71″	0′ 47.89″	1.76%	2.135%	—0′ 12.97″	+2′ 2.2875″
			Right			0 45 20.87	0 41.77	1.53		—0 1.81	
		2	Left	17.521 mm.	0 48 56.39	0 53 20.74	1 21.33	2.54		+4 24.35	
			Right			0 52 55.97	1 26.10	2.71		+3 59.58	
	P.	1	Left	16.245 mm.	0 45 22.68	0 45 47.6	0 47.525	1.72	1.7325	+0 24.92	+0 51.04
			Right			0 45 34.28	0 45.31	1.66		+0 11.60	
		2	Left	17.521 mm.	0 48 56.39	0 50 33.89	0 50.515	1.66		+1 37.50	
			Right			0 50 6.53	0 56.93	1.89		+1 10.14	

TABLE III.

Set.	Observer.	Series.	Eye.	Measurement of Normal Stimulus.	Visual Angle of Normal Stimulus.	Estimated Value.	Mean Variation.	% of Estimated Value.	Average %.	Constant Error.	Average of Constant Errors
V	McC.	1	Left	12.6 mm.	35′ 11.92″	36′ 2.29″	55.57″	2.57%	2.105%	+0′ 50.37″	+23.615″
			Right			36 16.07	43.35	1.99		+1 4.15	
		2	Left	11.601 mm.	32 24.53	32 11.99	37.78	1.96		—0 12.54	
			Right			32 17.01	36.90	1.90		—0 7.52	
	P.	1	Left	12.6 mm.	35 11.92	36 13.83	40.845	1.88	1.5775	+1 1.91	+25.2375
			Right			35 43.92	40.205	1.88		+0 32.00	
		2	Left	11.601 mm.	32 24.53	32 40.38	27.22	1.38		+0 15.85	
			Right			32 17.42	22.545	1.17		—0 9.81	
VI	McC.	1	Left	8.289 mm.	23′ 9.44″	22′ 26.27″	17.02″	1.26%	2.1125%	—0′ 43.17″	+37.7975″
			Right			22 31.08	17.37	1.29		—0 38.36	
		2	Left	9.291 mm.	25 57.38	27 42.59	42.81	2.58		+1 45.21	
			Right			28 4.89	55.94	3.32		+2 7.51	
	P.	1	Left	8.289 mm.	23 9.44	22 47.84	18.725	1.37	1.985	—0 21.60	+46.0175
			Right			22 52.48	18.665	1.36		—0 16.96	
		2	Left	9.291 mm.	25 57.38	27 55.69	43.12	2.57		+1 58.31	
			Right			27 41.7	43.89	2.64		+1 44.32	

SUMMARY TABLE.

Set.	Series.	Measurement of Normal Discs.	Average.	Average %.	
				J. McCrea.	H.J.Pritch'd.
I	1	35.902 mm.			
	2	37.306 mm.	36.604 mm.	1.3125	1.0975
II	1	30.155 mm.			
	2	28.958 mm.	29.5565 mm.	1.2975	.975
III	1	22.73 mm.			
	2	23.923 mm.	23.3265 mm.	1.3925	1.1775
IV	1	16.245 mm.			
	2	17.521 mm.	16.883 mm.	2.135	1.7325
V	1	12.6 mm.			
	2	11.601 mm.	12.1005 mm.	2.105	1.5775
VI	1	8.289 mm.			
	2	9.291 mm.	8.79 mm.	2.1125	1.985

The order in which these sets were observed is as follows: IV, V, III, II, VI, I.

subtended by that diameter. In the 7th appears the value of the visual angle, when it was judged to be equal to the normal stimulus. The 8th and 9th columns will be understood from what has already been said in this paper. By Constant Error is meant the difference between the estimated value and the normal magnitude.

The sets in the table are arranged according to the relative sizes of the discs used, beginning at the largest, and not according to the order in which the observations were made.

In a comparison of the results which are given in the Summary Table, it will be noticed that the percentages, although not nearly equal, still show a certain approximation to constancy. Whilst the normal magnitudes vary between extremes, which are to each other about 1:4, the per cent. of the average error varies for Observer M. only between the limits of $1:1\frac{2}{3}$ and for Observer P., 1:2. Thus the Law of Weber does not seem to hold exactly for surface magnitudes; but the results show a decided approach towards it. It will be observed, however, that with a certain degree of

regularity the percentage increases as the magnitude of the object decreases. But we see that the percentage of the mean variation in Set IV is greater than this regularity of increase would demand. This may be accounted for by the fact that this was the first set of observations made, and the accuracy of judging has probably increased with a year's practice. In Set II, where the average stimulus had a visual angle of about 1° 20′, the mean variation was the smallest; and as the visual angle decreased in size from that magnitude, the mean variation increased, as is shown in the results obtained from Sets III, IV, V, and VI. It was noticed, during the course of the observations, that as the size of the disc decreased, the irradiation of light became greater. The irradiation was so great in Set VI, where the visual angle of the normal stimulus was about 0° 24′, that it was found necessary to put two additional sheets of tissue paper over the open end of the case; and even then the irradiation had a disturbing influence upon the judgments. This increase in the irradiation of light may account in whole or in part for the increase of the mean variation as the magnitude of the stimulus decreases.

Only one set of experiments was made in which the normal stimulus was greater than 1° 20′, and in this set the mean variation was greater than the mean variation in Set II. There has not been a sufficient number of experiments performed to warrant one in coming to any definite conclusion as to the cause of this irregularity. It may, however, be due to the fact that the visual angle is so large that the eye in observing is inclined to make movements which are so great as to interfere with accuracy in judging. Beyond this suggestion nothing further can be stated at present as to the cause of this irregularity.

In examining the Summary Table one will notice a marked correspondence between the results obtained by the two observers. With the exception of Set IV, which was the first made, when the mean variation increases for the one it also increases for the other. Throughout the whole six sets the mean variation for Observer P. has been considerably less than for Observer M.; but on comparison it is found that they bear a comparatively constant relation to each other. In Set I the mean variation for Observer P. is 83.6% of the mean variation for Observer M.; in Set II, 75.1% ; in Set III, 84.5% ; in Set IV, 81.1% ; in Set V, 72.08%, and in Set VI, 93.9% ; the average of these is 81.7%.

In addition to what has been said, it is of interest to consider the Constant Error, as seen in the tables. One of the results of Mr. Quantz's experiments was that the movable

disc was always underestimated. This fact, which as yet cannot be accounted for, is decidedly confirmed by these trials. A glance over the last column of the tables, which contains the total averages of the Constant Errors for the different sets, will show that there is, in nearly all cases, a *positive* deviation. This means that, in general, the moved disc was decidedly underestimated.

Nevertheless, on examining the different series of each set separately, it is found that when the variable stimulus is closer to the observer than the normal stimulus, it is judged to be equal to the normal when its visual angle is greater than that of the normal, *i. e.*, it is underestimated. But when the variable stimulus is farther away from the observer than the normal stimulus, it is judged to be equal to the normal when its visual angle is less than that of the normal, *i. e.*, it is overestimated. As may be seen by an examination of the tables, this is constant for both observers throughout almost the whole of the six sets. We may, therefore, conclude, with a considerable degree of certainty, that when we compare the size of two objects lying at different distances, the nearer object is underestimated or the more distant one is overestimated. Although it was the desire in these experiments to take no account of distance and to attend only to the size of the objects observed, it is evident that the observers' knowledge of the distance has had a slight influence on their judgment. Probably the underestimation of the nearer object may be accounted for by the fact that since it is known to be nearer to the eye of the observer than the normal stimulus, it is expected that it will appear larger, and hence it is judged to be smaller than it really is. And in the same way, when the variable is at a greater distance from the observer than the normal, it is expected that it will appear smaller, and hence it is judged to be larger than it really is. It may be mentioned here that these results do not correspond with those obtained by Goetz Martius.[1]

However, the discrepancy may rest upon unforeseen circumstances, which may be ascertained by future research. In the experiments of Goetz Martius, the objects were seen successively, thus making necessary a change of the convergence and accommodation (all of the observations of Goetz Martius were made binocularly, while in these experiments the observations were made monocularly, and one of the objects was always at rest at the same normal distance). In all likelihood the regularity with which the positive and nega-

[1] *Philos. Stud.*, V, p. 601 ff.

tive errors appear in these trials has something to do with the incongruence of the visual angle and the angle of regard; the so-called Parallax of Indirect Vision, to which Kirschmann attributes so great a significance for the monocular depth perception. However, at present the connection is not clearly seen, but the reader, who is interested in the matter, is referred to the articles of this author on "The Parallax of Indirect Vision and the Slit-formed Pupil of the Cat"[1] and "The Metallic Lustre."[2]

It must be remembered that the trials of Goetz Martius, which were concerned with linear magnitudes, had entirely different aims and methods; and, therefore, the results must, to a certain extent, be incomparable with ours. His method was adapted chiefly to the problem whether there is overestimation or underestimation in different distances, and not to find out the accuracy of our judgment for the magnitudes themselves.

How great the accuracy of the judgment for surface magnitudes is may be recognized by regarding the diagrams of Figure 2, in which are given the magnitudes of the discs used as nearly as they can be reproduced in a drawing. The left ones represent the larger, and the right ones represent the smaller of each of the six pairs. (They are arranged concentrically in order to save space.) Before the diagram was drawn it was proposed to represent the accuracy of the judgment, *i. e.*, the mean variation, by the thickness of the stroke representing the circumference; but a simple calculation from the tables will show that the thickness of the stroke would have to vary between about $\frac{1}{5}$ mm. (0.234 for M. and 0.198 for P.) for the greatest circle, and about $\frac{1}{10}$ mm. (0.098 for M. and 0.092 for P.) for the smallest circle. These magnitudes are too small to be well represented in a drawing, and certainly they cannot be represented in a cut. Thus the accuracy of the judgment in these trials on surface magnitudes is too great to be indicated by the thickness of the stroke.

The results of these experiments could be summarized in the following propositions :—

1. The accuracy of our judgment of surface magnitudes is astonishingly great. The mean variation for visual angles of $1°$ or less was always below $1'$ in magnitude, and for angles up to $1° 45'$ in magnitude, it never exceeded $1' 20''$.

2. Although the results do not entirely fulfill the demands of the Psychophysical Law, yet they show a certain approximation towards it.

[1] *Philos. Stud.*, Vol. IX, pp. 447-495.
[2] *Ibid.*, Vol. XI, pp. 147-189.

3. In the comparison of a fixed object with one which is moved towards or from the eye, the latter is overestimated when it is at a greater distance from the eye, and underestimated when it is at a lesser distance. Taking all of the results together, the moved object is decidedly underestimated. This latter fact confirms fully the results of the earlier investigation of Mr. Quantz.